Clase 3: El Teorema de Lax

Métodos Numéricos para Ecuaciones Diferenciales Parciales CM032

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30 de marzo de 2026

Teorema de Equivalencia de Lax

Primero presentamos un marco abstracto. Sea $(V, \|\cdot\|)$ un espacio de Banach, $V_0 \subset V$ un subespacio denso de V. Sea $L\colon V_0 \subset V \to V$ un operador lineal. El operador L suele ser no acotado y puede pensarse como un operador diferencial. Considere el problema del valor inicial

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu & \text{en } [0, T] . \\ u = u_0 & \text{en } \{t = 0\} . \end{cases} \tag{1}$$

Este problema también representa un problema de valor inicial / frontera con condiciones de valor de frontera homogéneo cuando ellas se incluyen en las definiciones del espacio V y el operador L. La siguiente definición da el significado de solución del problema.

Definición 1

Una función $u\colon [0, T] \to V$ es una solución del problema de valor inicial (6.2.1) si y solo si

$$\forall t \in [0, T] , u(t) \in V_0 : \lim_{\Delta t \to 0} \left\| \frac{1}{\Delta t} [u(t + \Delta t) - u(t)] - Lu(t) \right\| = 0 \text{ y } u(0) = u_0.$$

En la definición anterior, se entiende que el límite en (6.2.2) es el límite derecho en $t = 0$ y el límite izquierdo en $t = T$.

Definición 2

El problema de valor inicial (6.2.1) es bien planteado si para cualquier $u_0 \in V_0$, existe una solución única $u = u(t)$ y la solución depende continuamente del valor inicial: Existe una constante $c_0 > 0$ tal que si $u(t)$ y $\overline{u}(t)$ son las soluciones para los valores iniciales $u_0, u_0 \in V_0$, entonces

$$\sup_{t\in[0,T]} \|u(t) - \overline{u}(t)\|_V \leq c_0 \|u_0 - \overline{u}_0\|_V.$$

De ahora en adelante, suponemos que el problema del valor inicial (6.2.1) está bien planteado. Denotamos la solución como

$$u(t) = S(t)\, u_0, \quad u_0 \in V_0, \quad t \in [0, T]$$

Usando la linealidad del operador L, es fácil ver que el operador solución $S(t)$ es lineal. De la propiedad de dependencia continua (6.2.3), tenemos

$$\sup_{t\in[0,T]} \|S(t)[u_0 - \overline{u}_0]\|_V \leq c_0 \|u_0 - \overline{u}_0\|_V.$$

$$\forall u_0 \in V_0 : \sup_{t\in[0,T]} \|S(t)\, u_0\|_V \leq c_0 \|u_0 -\|_V.$$

Según el teorema 2.4.1, el operador $S(t) : V_0 \subset V \to V$ puede ser únicamente extendido a un operador lineal continuo $S(t) : V \to V$ con

$$\sup_{t\in[0,T]} \|S(t)\|_V \leq c_0.$$

Definición 3

For $u_0 \in V \setminus V_0$, we call $u(t) = S(t)\, u_0$ the generalized solution of the initial value problem (6.2.1).

Ejemplo 1

We use the following problem and its finite difference approximations to illustrate the use of the abstract framework of the section:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} & \text{en } [0,\pi] \times (0,T)\,. \\ u = u_0 & \text{en } [0,\pi] \times \{t = 0\}\,. \end{cases}$$

We take $V = C_0[0,\pi] = \{v \in C[0,\pi] \mid v(0) = v(\pi) = 0\}$ with the norm $\|\cdot\|_{C[0,\pi]}$. We choose

$$V_0 = \left\{ v \mid v(x) = \sum_{j=1}^{n} a_j \sin(jx)\,, a_j \in \mathbb{R}, n \in \mathbb{N} \right\}$$

The verification that V_0 is dense in V is left as an exercise.

If $u_0 \in V_0$, then for some integer $n \geq 1$ and $b_1, \ldots, b_n \in \mathbb{R}$,

$$u_0(x) = \sum_{j=1}^{n} b_j \sin(jx)\,.$$

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For this u_0, it can be verified directly that the solution is

$$u(x,t) = \sum_{j=1}^{n} b_j \exp\left(-\nu j^2 t\right) \sin(jx).$$

By using the maximum principle for the heat equation (see, e.g. [78] or other textbooks on partial differential equations),

$$\min\left\{0, \min_{x\in[0,\pi]} u_0(x)\right\} \leq u(x,t) \leq \max\left\{0, \max_{x\in[0,\pi]} u_0(x)\right\}$$

we see that

$$\forall t \in [0,T] : \max_{x\in[0,\pi]} |u(x,t)| \leq \max_{x\in[0,\pi]} |u_0(x)|.$$

Thus the solution operator $S(t) : V_0 \subset V \to V$ is bounded. Then for a general $u_0 \in V$, the problem (6.2.4) has a unique solution. If $u_0 \in V$ has a piecewise continuous derivative in $[0,\pi]$, then from the theory of Fourier series,

$$u_0(x) = \sum_{j=1}^{\infty} b_j \sin(jx), \quad \text{where } b_j = \frac{2}{\pi}\int_0^{\pi} u_0(x)\sin(jx)\,dx.$$

and the solution $u(t)$ can be expressed as

$$u(x,t) = S(t)\,u_0(x) = \sum_{j=1}^{\infty} b_j \exp\left(-\nu j^2 t\right)\sin(jx).$$

Return to the abstract problem (6.2.1). We present two results, the first one is on the time continuity of the generalized solution and the second one shows the solution operator $S(t)$ forms a semigroup.

Proposición 1

For any $u_0 \in V$, the generalized solution of the initial value problem (6.2.1) is continuous in t.

Demostración.

For any $u_0 \in V$, there exists a sequence $\{u_{0n}\} \subset V_0$ such that

$$\lim_{n\to\infty} \|u_{0n} - u_0\|_V = 0.$$

Since for any $n \in \mathbb{N}$, $u_n(t) = S(t)\, u_{0n}$ is continuous in t, the generalized solution $u(t) = S(t)\, u_0$ is also continuous in t. The proof is complete. $\square$

Proposición 2

Assume the problem (6.2.1) is well-posed. Then for all $t_1, t_0 \in [0, T]$ such that $t_1 + t_0 \leq T$, we have $S(t_1 + t_0) = S(t_1) S(t_0)$.

Demostración.

For any $u_0 \in V_0$, we have

$$S(t_1 + t_0) u_0 = S(t_1) [S(t_0) u_0] .$$

Since V_0 is dense in V and both $S(t_1 + t_0)$ and $S(t_1) S(t_0)$ are continuous linear operators on V, we have

$$S(t_1 + t_0) = S(t_1) S(t_0) .$$

The proof is complete. □

Now we introduce a finite difference method defined by a one-parameter family of uniformly bounded linear operators

$$C(\Delta t) : V \to V, \quad 0 < \Delta t \leq \Delta_0.$$

Here $\Delta_0 > 0$ is a fixed number. The family $\{C(\Delta t)\}_{0<\Delta t\leq\Delta_0}$ is said to be uniformly bounded if there is a constant c such that

$$\forall \Delta t \in (0, \Delta_0] : \|C(\Delta t)\| \leq c$$

The approximate solution is then defined by

$$u_{\Delta t}(m\Delta t) = C(\Delta t)^m u_0, \quad m = 0, 1, \ldots, \lfloor T/\Delta t \rfloor.$$

Definición 4

The difference method is consistent if there exists a dense subspace V_c of V such that for all $u_0 \in V_c$, for the corresponding solution u of the initial value problem (6.2.1), we have

$$\lim_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left[C(\Delta t)\, u(t) - u(t+\Delta t) \right] \right\| = 0 \text{ uniformly in } [0, T].$$

Assume $V_c \cap V_0 \neq \emptyset$. For $u_0 \in V_c \cap V_0$, we write

$$\frac{1}{\Delta t} \left[C(\Delta t)\, u(t) - u(t+\Delta t) - u(t+\Delta t) \right] = \left[\frac{C(\Delta t) - I}{\Delta t} - L \right] u(t) - \left[\frac{u(t+\Delta t) - u(t)}{\Delta t} - Lu(t) \right].$$

Since

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} \to u'(t) \quad \text{as } \Delta t \to 0,$$

we have

$$\lim_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \left[C(\Delta t)\, u(t) - u(t+\Delta t) \right] \right\| = 0.$$

Ejemplo 2

Let us now consider the forward method and the backward method from Example 6.1.1 for the sample problem (6.2.4). For the forward method, we define the operator $C(\Delta t)$ by the formula

$$C(\Delta t)\, v(x) = (1 - 2r)\, v(x) + r \left[v(x + \Delta x) + v(x - \Delta x) \right],$$

where $\Delta x = \sqrt{\frac{\nu \Delta t}{r}}$ and if $x \pm \Delta x \in [0, \pi]$, then the function v is extended by oddness with period 2π.

We identify Δt with h_t and Δx with h_x. Then $C(\Delta t) : V \to V$ is a linear operator and it can be shown

$$\forall v \in V : \|C(\Delta t)\|_V \leq (|1-2r| + 2r)\,\|\|_V\,.$$

So

$$\|C(\Delta t)\| \leq |1-2r| + 2r,$$

and the family $\{C(\Delta t)\}_{0<\Delta t\leq \Delta_0}$ is uniformly bounded. The difference method is

$$u\Delta t\,(t_m) = C(\Delta t)\,u_{\Delta t}\,(t_{m-1}), \quad m = 1, 2, \ldots, \lfloor T/\Delta t \rfloor$$

or

$$u_{\Delta t}\,(\cdot, t_m) = C(\Delta t)^m u_0(\cdot)\,.$$

Notice that in this form, the difference method generates an approximate solution $u_{\Delta t}(x,t)$ that is defined for $x \in [0,\pi]$ and $t = t_m$, $m = 0, 1, \ldots, N_t$. Since

$$u_{\Delta t}\left(x_j, t_{m+1}\right) = (1-2r)\,u_{\Delta t}\left(x_j, t_m\right) + r\left[u_{\Delta t}\left(x_{j-1}, t_m\right) + u_{\Delta t}\left(x_{j+1}, t_m\right)\right], 1 \leq j \leq N_x - 1, 0 \leq m \leq N_t - 1,$$

we see that the relation between the approximate solution $u_{\Delta t}$ and the solution v defined by the ordinary difference scheme (6.1.8)–(6.1.10) (with $f_m^j = 0$) is

$$u_{\Delta t}\left(x_j, t_m\right) = v_m^j, \quad 0 \leq j \leq N_x, 0 \leq m \leq N_t.$$

As for the consistency, we take $V_c = V_0$. For the initial value function (6.2.6), we have the formula (6.2.7) for the solution which is obviously infinitely smooth. Now using Taylor expansions at (x,t), we have

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